ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2025, Vol. 86, No. 2, pp. 99–112.

Optimal Absolute Stabilization of Unknown Lurie Systems Based on Experimental Data and A Priori Information

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Received January 18, 2024
Revised August 20, 2024
Accepted November 26, 2024

Abstract—This paper considers Lurie systems composed of an unknown linear subsystem and unknown nonlinear functions belonging to given sectors. For such systems, a method is developed to design an optimal absolutely stabilizing control law based on experimental data and a priori information. The method involves the minimax approach in which an integral quadratic performance index is maximized at the intersection of two matrix ellipsoidal sets selected from experimental data and a priori information. The simulation results of a nonlinear oscillator show the advantage of the obtained control law over the classical robust controller designed based on a priori information.

Keywords: Lurie systems, uncertainty, robust control, experimental data, linear matrix inequalities (LMIs)

DOI: 10.31857/S0005117925020017

1. INTRODUCTION

Nonlinear Lurie systems consist of unknown linear subsystems and unknown nonlinear functions belonging to given sectors [1]. In this paper, we construct control laws for this class of systems using the control design methods for unknown linear dynamic systems developed in [2–4]. The absence of any mathematical model of the controlled system is compensated here by measurement data of the system trajectory on a finite horizon and some a priori information. In this case, the identification problem is not solved for the system, and the control objective is achieved even under non-identifiability conditions. These methods involve the minimax robust control design approach in which the guaranteed value of the performance index is found for any system from some uncertainty set. In contrast to classical robust controllers (e.g., see the review [5]), where this set is chosen based on only a priori information, here the uncertainty set is separated by using both experimental data and a priori information. As a result, one obtains a control law with a much smaller guaranteed value of the performance index; see below.

In recent time, the use of experimental information for the direct design of control laws has received much attention; for example, see [6–8] and the bibliography provided in [2–4]. These research works mainly addressed control laws for linear systems. To the authors' best knowledge, the paper [9] is currently the only one devoted to experimental data-based control design for systems with a nonlinear vector function satisfying a quadratic inequality. According to the assumptions made in [9], there are no disturbances during the experiment and the control law ensures the so-called persistent excitation condition, necessary for the identifiability of the unknown parameters. Furthermore, the dimension of the variables in the linear matrix inequalities (LMIs) for computing the controller parameters grows with the number of measurements, which complicates control law implementation. In contrast, the approach discussed below covers systems with several nonlinear sector functions, considers disturbances during the experiment, and requires no system identifiability; in addition, the dimensions of the variables are determined only by those of the state and control vectors and do not depend on the number of measurements.

2. PROBLEM STATEMENT

Consider a controlled uncertain nonlinear Lurie system consisting of a linear subsystem of the form

$$\begin{aligned} \partial x(t) &= Ax(t) + Bu(t) + Fv(t), \\ z(t) &= Cx(t) + Du(t) \end{aligned} \tag{2.1}$$

closed by a nonlinear continuous vector function of the form

$$v(t) = \varphi(y(t), t), \quad y(t) = L^{\mathrm{T}} x(t), \tag{2.2}$$

with the following notations: ∂ stands for the differentiation operator in the continuous-time case or the shift operator in the discrete-time case; $x(t) \in \mathrm{R}^{n_x}$ is the state vector; $y(t) \in \mathrm{R}^{n_y}$ is the output; $u(t) \in \mathrm{R}^{n_u}$ is the control vector (input); $z(t) \in \mathrm{R}^{n_z}$ is the performance output; finally, $\varphi(y, t) \in \mathrm{R}^{n_y}$ is an unknown vector function such that $\varphi(0, t) \equiv 0$. For all $t \geqslant 0$, each component $\varphi_i(y_i, t)$ of the function (2.2) is located in a corresponding finite sector $[\alpha_i, \beta_i]$, i.e.,

$$\alpha_i \leqslant \frac{\varphi_i(y_i, t)}{y_i} \leqslant \beta_i, \quad i = 1, \ldots, n_y. \tag{2.3}$$

By assumption, the system matrices A, B, and F are unknown and the initial state $x(0) = x_0$ is uncertain. The problem statement will be further refined; generally speaking, it is required to design linear state-feedback controllers $u(t) = \Theta x(t)$ based on a priori information and experimental data under which the closed loop system will be absolutely stable, i.e., the trivial equilibrium $x = 0$ of system (2.1), (2.2) will be asymptotically stable for all functions $\varphi(y, t)$ from the specified class, and the transient will satisfy the following upper bound under arbitrary initial conditions:

$$\sup_{x_0 \neq 0} \frac{\|z\|^2}{x_0^{\mathrm{T}} R^{-1} x_0} < \gamma^2, \tag{2.4}$$

where $R = R^{\mathrm{T}} > 0$ is a weight matrix and $\|\xi\|^2 = \sum_{t=0}^{\infty} |\xi(t)|^2$ in the continuous-time case or $\|\xi\|^2 = \int_{t=0}^{\infty} |\xi(t)|^2 \, dt$ in the discrete-time case.

3. EXPERIMENTAL DATA AND A PRIORI INFORMATION

Information about the unknown parameters of system (2.1) is extracted from a finite set of measurements of its trajectory. Let a disturbance $w(t)$ affect the system during an experiment so that the system equations take the form

$$\begin{aligned} \partial x(t) &= Ax(t) + Bu(t) + Fv(t) + B_w w(t), \\ z(t) &= Cx(t) + Du(t). \end{aligned} \tag{3.1}$$

Suppose that in the experiment preceding the control design procedure, it is possible to measure the values of the system's nonlinear function belonging to given sectors. In the discrete-time case, there are available measurements of the state and nonlinear function, $x_0, x_1, \ldots, x_N$ and

$\varphi(y_0, 0), \ldots, \varphi(y_{N-1}, N-1)$, respectively, under chosen controls $u_0, \ldots, u_{N-1}$ and some unknown disturbances $w_0, \ldots, w_{N-1}$. We compile the matrices

$$X = (x_0 \cdots x_{N-1}), \quad X_+ = (x_1 \cdots x_N), \quad U = (u_0 \cdots u_{N-1}),$$
$$\Phi = (\varphi(y_0, 0) \cdots \varphi(y_{N-1}, N-1), \quad W = (w_0 \cdots w_{N-1}).$$

In the continuous-time case, there are available measurements of the state, its derivative, and nonlinear function, $x(t_0), \ldots, x(t_{N-1})$, $\dot{x}(t_0), \ldots, \dot{x}(t_{N-1})$, and $\varphi(y(t_0), t_0), \ldots, \varphi(y(t_{N-1}), t_{N-1})$, respectively, at time instants $t_0, \ldots, t_{N-1}$ under chosen controls $u(t_0), \ldots, u(t_{N-1})$ and some unknown disturbances $w(t_0), \ldots, w(t_{N-1})$. We compile the matrices

$$X = (x(t_0) \cdots x(t_{N-1})), \quad X_+ = (\dot{x}(t_0) \cdots \dot{x}(t_{N-1})), \quad U = (u(t_0) \cdots u(t_{N-1})),$$
$$\Phi = (\varphi(y(t_0), t_0) \cdots \varphi(y(t_{N-1}), t_{N-1})), \quad W = (w(t_0) \cdots w(t_{N-1})).$$

In both cases, the experimental data matrices satisfy the relations

$$X_+ = A_{real} X + B_{real} U + F_{real} \Phi + B_w W, \tag{3.2}$$

where A_{real}, B_{real}, and F_{real} are the real unknown system matrices. With the notations

$$\Delta_{real} = (A_{real}\, B_{real}\, F_{real}), \quad \widehat{X} = \operatorname{col}(X, U, \Phi),$$

equations (3.2) can be written as the linear matrix regression

$$X_+ = \Delta_{real} \widehat{X} + \widehat{W}, \quad \widehat{W} = B_w W. \tag{3.3}$$

Assume that the disturbances, which also include the approximate calculation errors of the derivatives, satisfy the condition

$$\widehat{W} \widehat{W}^{\mathrm{T}} \leqslant \Omega. \tag{3.4}$$

In particular, if $\|w(t)\|_\infty \leqslant d_w$ for all t and a given value d_w (the error level), then $\Omega = d_w^2 n_w N B_w B_w^{\mathrm{T}}$. In the case $\sum_{i=0}^{N-1} |w(t_i)|^2 \leqslant \nu^2$ (the total "energy" of the disturbances is bounded above during the experiment), we have $\Omega = \nu^2 B_w B_w^{\mathrm{T}}$.

Let us define the set $\mathbf{\Delta_p}$ of all matrices Δ of dimensions $n_x \times (n_x + n_u + n_y)$ that could generate the experimental matrices Φ, Φ_+, and Z under the chosen controls U and some admissible errors $\widehat{W}$ satisfying the constraint (3.4). For these matrices, equality (3.3) must hold under some matrix $\widehat{W}$ satisfying (3.4). Hence,

$$\mathbf{\Delta_p} = \left\{ \Delta : \, X_+ = \Delta \widehat{X} + \widehat{W}, \quad \widehat{W} \widehat{W}^{\mathrm{T}} \leqslant \Omega \right\},$$

and $\Delta \in \mathbf{\Delta_p}$ iff

$$(X_+ - \Delta \widehat{X})(X_+ - \Delta \widehat{X})^{\mathrm{T}} \leqslant \Omega. \tag{3.5}$$

It is obvious that $\Delta_{real} \in \mathbf{\Delta_p}$. For further use, we represent this last inequality as

$$(\Delta \quad I)\, \Psi^{(1)}\, (\Delta \quad I)^{\mathrm{T}} \leqslant 0, \tag{3.6}$$

where the symmetric matrix $\Psi^{(1)}$ of order $2n_x + n_u + n_y$ is partitioned into appropriate blocks $\Psi_{ij}^{(1)}$, $i, j = 1, 2$, as follows:

$$\Psi^{(1)} = \left(\begin{array}{c|c} \widehat{X}\widehat{X}^{\mathrm{T}} & \star \\ \hline -X_+ \widehat{X}^{\mathrm{T}} & X_+ X_+^{\mathrm{T}} - \Omega \end{array} \right). \tag{3.7}$$

Thus, the set of all matrices Δ consistent with the experimental data satisfies inequality (3.6).

Generally speaking, the set $\mathbf{\Delta_p}$ is unbounded. To establish its boundedness conditions, we denote by $\mathrm{Im}(\cdot)$, $\mathrm{Ker}(\cdot)$, $\mathrm{span}(\cdot)$, and $\mathrm{rank}(\cdot)$ the image, kernel, linear column subspace, and column rank of an appropriate matrix, respectively. Under the assumption $\mathrm{rank}\,\widehat{X} = s \leqslant \min\{n_x + n_u + n_y, N\}$, the matrix $\widehat{X}$ admits the singular decomposition [10]

$$\widehat{X} = (M_1\, M_2) \begin{pmatrix} \Sigma & 0_{s\times(N-s)} \\ 0_{(n_x+n_u+n_y)\times s} & 0_{(n_x+n_u+n_y)\times(N-s)} \end{pmatrix} \begin{pmatrix} G_1^{\mathrm{T}} \\ G_2^{\mathrm{T}} \end{pmatrix} = M_1 \Sigma G_1^{\mathrm{T}},$$
$$M_1 \in \mathrm{R}^{(n_x+n_u+n_y)\times s},\ M_2 \in \mathrm{R}^{(n_x+n_u+n_y)\times(n_x+n_u+n_y-s)}, \quad M = (M_1\, M_2)\,, \tag{3.8}$$

where $\Sigma = \mathrm{diag}(\lambda_1, \ldots, \lambda_s) > 0$, λ_i are the eigenvalues of the information matrix $\widehat{X}\widehat{X}^{\mathrm{T}}$, $\mathrm{span}\, M_1 = \mathrm{Im}\,\widehat{X}$, $\mathrm{span}\, M_2 = \mathrm{Ker}\,\widehat{X}^{\mathrm{T}}$, $\mathrm{span}\, G_1 = \mathrm{Im}\,\widehat{X}^{\mathrm{T}}$, $\mathrm{span}\, G_2 = \mathrm{Ker}\,\widehat{X}$, and $M^{\mathrm{T}}M = I$. Choosing the orthonormal basis of the columns of the matrix M, we introduce the corresponding variables

$$\widehat{\Delta} = \Delta\,(M_1\, M_2) = \left(\widehat{\Delta}^{(1)}\, \widehat{\Delta}^{(2)}\right), \quad \widehat{\Delta}^{(1)} \in \mathrm{R}^{n_x\times s}, \quad \widehat{\Delta}^{(2)} \in \mathrm{R}^{n_x\times(n_x+n_u+n_y-s)}$$

and denote $\widehat{X}^{(1)} = M_1^{\mathrm{T}}\widehat{X} = \Sigma G_1^{\mathrm{T}}$. In the new variables, the linear matrix regression (3.3) takes the form

$$X_+ = \widehat{\Delta}^{(1)}_{real}\widehat{X}^{(1)} + \widehat{W}, \tag{3.9}$$

where the matrix $\widehat{X}^{(1)}$ of dimensions $(s \times N)$ has a full column rank, and $\widehat{\Delta}^{(1)}_{real}$ is the "projection" of the matrix $\widehat{\Delta}_{real}$ into the subspace $\mathrm{Im}\,\widehat{X}$, i.e., its rows $\widehat{\Delta}^{(1)}_{real}$ are the projections of the rows of the matrix $\widehat{\Delta}_{real}$ into the subspace $\mathrm{Im}\,\widehat{X}$.

Lemma 3.1. *The set $\mathbf{\Delta_p}$ of all matrices consistent with the experimental data $\widehat{X} = \mathrm{col}\,(X, U, \Phi)$ that satisfy* (3.8) *is an unbounded degenerate "matrix ellipsoid" given by*

$$(\widehat{\Delta}^{(1)} - \widehat{\Delta}^{(1)}_{LS})\Sigma^2(\widehat{\Delta}^{(1)} - \widehat{\Delta}^{(1)}_{LS})^{\mathrm{T}} \leqslant \Gamma, \quad \widehat{\Delta}^{(2)} \in \mathrm{R}^{n_x\times(n_x+n_u+n_y-s)}, \tag{3.10}$$

where

$$\Gamma = \Omega + X_+[\widehat{X}^{(1)\mathrm{T}}\Sigma^{-2}\widehat{X}^{(1)} - I]X_+^{\mathrm{T}} \geqslant 0, \tag{3.11}$$

and $\widehat{\Delta}^{(1)}_{LS} = X_+\widehat{X}^{(1)\mathrm{T}}\Sigma^{-2}$ is the least-squares estimate of the matrix $\widehat{\Delta}^{(1)}_{real}$ in (3.9).

Corollary 3.1. *The set $\mathbf{\Delta_p}$ is bounded iff the rank condition*

$$\mathrm{rank}\begin{pmatrix} X \\ U \\ \Phi \end{pmatrix} = n_x + n_u + n_y \tag{3.12}$$

holds. In this case, the set $\mathbf{\Delta_p}$ consists of the matrices given by inequality (3.10) *in which $\widehat{\Delta}^{(1)} = \widehat{\Delta}$ and $\widehat{\Delta}^{(1)}_{LS} = \widehat{\Delta}_{LS}$.*

The proof of Lemma 3.1 is provided in the Appendix. By this lemma, in the general case, only the projection $\widehat{\Delta}^{(1)}_{real}$ of the unknown matrix into the subspace $\mathrm{Im}\,\widehat{X}$ can be identified from the obtained data. Under the rank condition (3.12), the matrix Δ_{real} in (3.3) is identifiable, and the matrix ellipsoid and the set $\mathbf{\Delta_p}$ are bounded. Note that the rank condition (3.12) holds only if the number of measurements is not smaller than the sum of the dimensions of the state, output, and

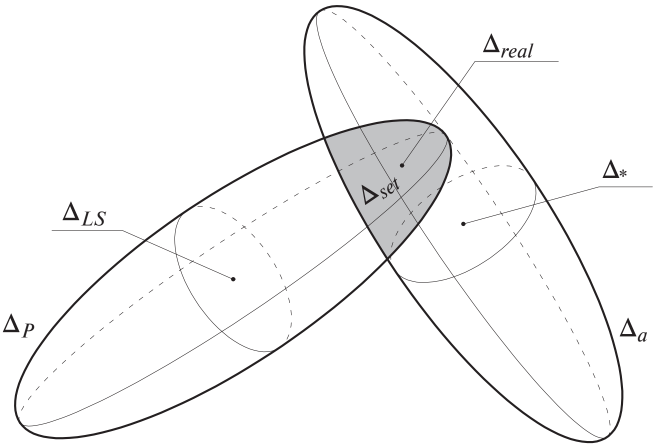

Fig. 1. The set $\boldsymbol{\Delta}_{\mathbf{set}}$ of all unknown parameters Δ consistent with experimental data and a priori information.

control vectors: $N \geqslant n_x + n_u + n_y$. The robust control design procedure considered in this paper does not require the rank condition, and the number of measurements can therefore be less than $n_x + n_u + n_y$.

Now let there exist additional information that the unknown matrix Δ_{real} satisfies the constraint

$$(\Delta - \Delta_*)(\Delta - \Delta_*)^{\mathrm{T}} \leqslant \rho^2 I, \quad \Delta_* = (A_*\, B_*\, F_*)\,, \tag{3.13}$$

where the matrix Δ_* contains the corresponding ones of the nominal system and the parameter ρ characterizes the size of the uncertainty domain. We write this inequality as

$$(\Delta \quad I)\,\Psi^{(2)}\,(\Delta \quad I)^{\mathrm{T}} \leqslant 0, \tag{3.14}$$

where the matrix $\Psi^{(2)}$ consists of the blocks $\Psi^{(2)}_{ij}$, $i, j = 1, 2$, and has the form

$$\Psi^{(2)} = \begin{pmatrix} I & | & \star \\ --- & --- & --- \\ -\Delta_* & | & \Delta_*\Delta_*^{\mathrm{T}} - \rho^2 I \end{pmatrix}. \tag{3.15}$$

We introduce the following notations: $\boldsymbol{\Delta}_{\mathbf{a}}$ is the set of all matrices Δ satisfying inequality (3.14), and $\boldsymbol{\Delta}_{\mathbf{set}} = \boldsymbol{\Delta}_{\mathbf{p}} \bigcap \boldsymbol{\Delta}_{\mathbf{a}}$ is the set of all matrices Δ satisfying inequalities (3.6) and (3.14). Obviously, $\Delta_{real} \in \boldsymbol{\Delta}_{\mathbf{set}}$. Figure 1 illustrates a possible arrangement of the sets $\boldsymbol{\Delta}_{\mathbf{p}}$ and $\boldsymbol{\Delta}_{\mathbf{a}}$ and their intersection $\boldsymbol{\Delta}_{\mathbf{set}}$.

In the current notations, the optimal absolute stabilization problem of the unknown Lurie system (2.1) can be formulated as follows: it is required to design, directly from input and state measurements, a state-feedback controller $u = \Theta x$ under which, for all systems whose matrices are consistent with the a priori information and experimental data and whose nonlinear functions belong to the given sectors (2.3), the closed loop system will be absolutely stable and the performance index $J(\Theta)$ will be bounded above by a given constant:

$$J(\Theta) = \sup_{\Delta \in \boldsymbol{\Delta}_{\mathbf{set}}} \sup_{\varphi(y,t)} \sup_{x_0 \neq 0} \frac{\|z\|^2}{x_0^{\mathrm{T}} R^{-1} x_0} < \gamma^2. \tag{3.16}$$

4. PRELIMINARY TRANSFORMATIONS AND AUXILIARY STATEMENTS

Consider the class of Lurie systems in which the components of the vector function $\varphi(y, t)$ satisfy condition (2.3). Before proceeding to the problem solution, we make some transformations to simplify the further presentation. Let us introduce a vector function $\widehat{\varphi}(y,t)$ with the components

$$\widehat{\varphi}_i(y_i,t) = \frac{1}{\beta_i - \alpha_i}[\varphi_i(y_i,t) - \alpha_i y_i], \quad i = 1,\dots,n_y. \tag{4.1}$$

Then equations (2.1) and (2.2) take the form

$$\begin{aligned} \partial x(t) &= (A + F\Lambda_1 L^{\mathrm{T}})x(t) + Bu(t) + F\Lambda_2\widehat{v}(t), \\ z(t) &= Cx(t) + Du(t), \end{aligned} \tag{4.2}$$

where $\Lambda_1 = \operatorname{diag}(\alpha_1,\dots,\alpha_{n_y})$, $\Lambda_2 = \operatorname{diag}(\beta_1 - \alpha_1,\dots,\beta_{n_y} - \alpha_{n_y})$, and

$$\widehat{v} = \widehat{\varphi}(y,t), \quad y = L^{\mathrm{T}}x, \tag{4.3}$$

while the functions $\widehat{\varphi}_i(y_i,t)$ satisfy the constraints (2.3) for $\alpha_i = 0$ and $\beta_i = 1$, $i = 1,\dots,n_y$, i.e., belong to the sector $[0,1]$.

We show that a Lyapunov function ensuring the absolute stability of the Lurie system with the guaranteed value of the quadratic performance index can be found by solving the corresponding worst-case disturbance problem in the linear system; for details, see [11]. Namely, for the Lurie system

$$\begin{aligned} \partial x(t) &= \mathcal{A}x(t) + \mathcal{F}v(t), \\ z(t) &= Cx(t), \end{aligned} \tag{4.4}$$

$$v(t) = \varphi(y(t),t), \quad y(t) = L^{\mathrm{T}}x(t), \tag{4.5}$$

with a stable matrix $\mathcal{A}$ and a vector function $\varphi(y,t)$ whose components lie in the sector $[0,1]$, we have the following result.

Lemma 4.1. *Assume that along the trajectories of the linear discrete- or continuous-time system* (4.4), *a function* $V(x) = x^{\mathrm{T}}Yx$ *with* $0 < Y = Y^{\mathrm{T}} < \gamma^2 R^{-1}$ *satisfies the inequality*

$$\triangle V + |z|^2 - v^{\mathrm{T}}\Gamma^{-1}(v - L^{\mathrm{T}}x) < 0, \quad \dot{V} + |z|^2 - v^{\mathrm{T}}\Gamma^{-1}(v - L^{\mathrm{T}}x) < 0, \tag{4.6}$$

where $\Gamma = \operatorname{diag}(\gamma_1,\dots,\gamma_{n_y}) > 0$, *for all* x *and* v $(|x|^2 + |v|^2 \neq 0)$. *Then the function* $V(x)$ *ensures the absolute stability of the Lurie system* (4.4), (4.5) *and, in addition,* $\|z\|^2 < \gamma^2 x_0^{\mathrm{T}} R^{-1} x_0$.

Remark 1. With the change of variables $\widehat{v} = \Gamma^{-1/2}(v - \frac{1}{2}L^{\mathrm{T}}x)$ and the performance output $\widehat{z} = \operatorname{col}(C, \frac{1}{2}\Gamma^{-1/2}L^{\mathrm{T}})x$, equations (4.4) become

$$\begin{aligned} \partial x(t) &= \left(\mathcal{A} + \frac{1}{2}\mathcal{F}L^{\mathrm{T}}\right)x(t) + \mathcal{F}\Gamma^{1/2}\widehat{v}(t), \\ \widehat{z}(t) &= \begin{pmatrix} C \\ \frac{1}{2}\Gamma^{-1/2}L^{\mathrm{T}} \end{pmatrix} x(t), \end{aligned} \tag{4.7}$$

where $\mathcal{A} + \frac{1}{2}\mathcal{F}L^{\mathrm{T}}$ is a Hurwitz matrix, and inequality (4.6) turns into $\dot{V} + |\widehat{z}|^2 - |\widehat{v}|^2 < 0$. Due to $Y < \gamma^2 R^{-1}$, this inequality is equivalent to the condition

$$\sup_{x_0,\widehat{v}} \frac{\|\widehat{z}\|^2}{x_0^{\mathrm{T}}\gamma^2 R^{-1}x_0 + \|\widehat{v}\|^2} < 1.$$

This means that the generalized H_∞ norm from the input $\widehat{v}$ to the output $\widehat{z}$ of system (4.7) with the weight matrix $\gamma^{-2}R$ is smaller than 1. As is easily checked, in the case of one nonlinearity ($z \equiv 0$), the resulting frequency condition $\|H\|_\infty < 1$ is equivalent to the circle criterion for absolute stability [12].

The next auxiliary statement characterizes the generalized H_∞ norm of a linear stable system

$$\begin{aligned} \partial x(t) &= \mathcal{A}x(t) + \mathcal{B}v(t), \\ z(t) &= \mathcal{C}x(t) \end{aligned} \tag{4.8}$$

in terms of the dual system.

Lemma 4.2 [3]. *The generalized H_∞ norm of system* (4.8) *with a weight matrix $\mathcal{R} > 0$ is smaller than* 1 *iff there exists a positive definite quadratic form $V_d(x_d) = x_d^{\mathrm{T}} P x_d$ with $P > \mathcal{R}$ such that*

$$\triangle V_d + |z_d|^2 - |v_d|^2 < 0, \quad \dot{V}_d + |z_d|^2 - |v_d|^2 < 0 \tag{4.9}$$

along the trajectories of the dual system

$$\begin{aligned} \partial x_d(t) &= \mathcal{A}^{\mathrm{T}} x_d(t) + \mathcal{C}^{\mathrm{T}} v_d(t), \\ z_d(t) &= \mathcal{B}^{\mathrm{T}} x_d(t) \end{aligned} \tag{4.10}$$

for all x_d and v_d ($|x_d|^2 + |v_d|^2 \neq 0$).

Remark 2. The matrices of the quadratic forms $V(x) = x^{\mathrm{T}} Y x$ and $V_d(x_d) = x_d^{\mathrm{T}} P x_d$ of the primal and dual systems are related by $P = Y^{-1}$.

Summarizing the above auxiliary statements and remarks, we arrive at the following result.

Theorem 4.1. *The closed-loop Lurie system* (2.1)–(2.3) *with given matrices A, B, and F and the state-feedback controller $u = \Theta x$ is absolutely stable and the performance index* (2.4) *is bounded above by a given constant γ^2 if the generalized H_∞ norm from the input v to the output z of the linear system*

$$\begin{aligned} \partial x(t) &= (A + B\Theta + F\Lambda L^{\mathrm{T}})x(t) + F\Lambda_2 \Gamma^{1/2} v(t), \\ z(t) &= \begin{pmatrix} C + D\Theta \\ \frac{1}{2}\Gamma^{-1/2} L^{\mathrm{T}} \end{pmatrix} x(t), \end{aligned} \tag{4.11}$$

with the weight matrix $\gamma^{-2}R$ and $\Lambda = \Lambda_1 + \frac{1}{2}\Lambda_2$ is smaller than 1.

Corollary 4.1. *In view of Lemma 4.2, the closed-loop Lurie system* (2.1)–(2.3) *is absolutely stable and the upper bound* (2.4) *holds if there exists a function $V(x_d) = x_d^{\mathrm{T}} P x_d$ with $P = P^{\mathrm{T}} > \gamma^{-2} R$ satisfying inequality* (4.9) *along the trajectories of the linear system*

$$\begin{aligned} \partial x_d(t) &= (A + B\Theta + F\Lambda L^{\mathrm{T}})^{\mathrm{T}} x_d(t) + \begin{pmatrix} C + D\Theta \\ \frac{1}{2}\Gamma^{-1/2} L^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} v_d(t), \\ z_d(t) &= \Gamma^{1/2} \Lambda_2 F^{\mathrm{T}} x_d(t). \end{aligned} \tag{4.12}$$

For system (4.12), inequality (4.9) can be written as an LMI, and we get another result.

Corollary 4.2. *System* (2.1)–(2.3) *with given matrices* A, B, *and* F *is absolutely stable and the upper bound* (2.4) *holds under the control law* $u = \Theta x$, *where* $\Theta = QP^{-1}$ *and* $P = P^{\mathrm{T}} > 0$, Q, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_{n_y}) > 0$, *and* $\gamma^2 > 0$ *satisfy the following LMIs:*

—*in the discrete-time case,*

$$\begin{pmatrix} -P & \star & \star & \star \\ AP + BQ + F\Lambda L^{\mathrm{T}}P & -P + F\Lambda_2\Gamma\Lambda_2F^{\mathrm{T}} & \star & \star \\ CP + DQ & 0 & -I & \star \\ \frac{1}{2}L^{\mathrm{T}}P & 0 & 0 & -\Gamma \end{pmatrix} < 0, \tag{4.13}$$

$$\begin{pmatrix} P & \star \\ I & \gamma^2R^{-1} \end{pmatrix} > 0;$$

—*in the continuous-time case,*

$$\begin{pmatrix} AP + PA^{\mathrm{T}} + BQ + Q^{\mathrm{T}}B^{\mathrm{T}} + F\Lambda L^{\mathrm{T}}P + PL\Lambda F^{\mathrm{T}} + F\Lambda_2\Gamma\Lambda_2F^{\mathrm{T}} & \star & \star \\ CP + DQ & -I & \star \\ \frac{1}{2}L^{\mathrm{T}}P & 0 & -\Gamma \end{pmatrix} < 0, \tag{4.14}$$

$$\begin{pmatrix} P & \star \\ I & \gamma^2R^{-1} \end{pmatrix} > 0.$$

5. THE DESIGN PROCEDURE OF OPTIMAL ABSOLUTELY STABILIZING CONTROLLERS

Now we write the equations of the closed-loop unknown system (2.1)–(2.3) in the form

$$\begin{aligned} \partial x(t) &= (A + B\Theta + F\Lambda_1L^{\mathrm{T}})x(t) + F\Lambda_2\widehat{v}(t), \\ z(t) &= (C + D\Theta)x(t), \end{aligned} \tag{5.1}$$

$$\widehat{v} = \widehat{\varphi}(y, t), \quad y = L^{\mathrm{T}}x, \tag{5.2}$$

where the components of the vector function $\widehat{\varphi}(y, t)$ are given by (4.1) and belong to the sector $[0, 1]$. In the theorem below, the parameters of the linear state-feedback controllers ensuring the absolute stability of the unknown nonlinear Lurie system and the guaranteed value of the performance index are expressed in terms of experimental data and a priori information.

Theorem 5.1. *The Lurie system* (2.1)–(2.3) *with the state-feedback controller* $u = \Theta x$ *is absolutely stable and the performance index* (3.16) *is bounded above,* $J(\Theta) < \gamma^2$, *if* $\Theta = QP^{-1}$, *where* $P = P^{\mathrm{T}} > 0$, Q, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_{n_y}) > 0$, $\gamma^2 > 0$, $\mu_1 \geqslant 0$, *and* $\mu_2 \geqslant 0$ *satisfy the following LMIs:*

—*in the discrete-time case,*

$$\begin{pmatrix} -P & \star & \star & \star \\ \begin{pmatrix} P \\ Q \\ \Lambda L^{\mathrm{T}} P \end{pmatrix} & \widehat{\Lambda}_2 \Gamma \widehat{\Lambda}_2^{\mathrm{T}} - \sum_{k=1}^{2} \mu_k \Psi_{11}^{(k)} & \star & \star \\ 0 & -\sum_{k=1}^{2} \mu_k \Psi_{21}^{(k)} & -P - \sum_{k=1}^{2} \mu_k \Psi_{22}^{(k)} & \star \\ \begin{pmatrix} C & D & 0 \\ 0 & 0 & \frac{1}{2} I \end{pmatrix} \begin{pmatrix} P \\ Q \\ L^{\mathrm{T}} P \end{pmatrix} & 0 & 0 & -\begin{pmatrix} I & \star \\ 0 & \Gamma \end{pmatrix} \end{pmatrix} < 0, \tag{5.3}$$

$$\begin{pmatrix} P & \star \\ I & \gamma^2 R^{-1} \end{pmatrix} > 0;$$

—*in the continuous-time case,*

$$\begin{pmatrix} \widehat{\Lambda}_2 \Gamma \widehat{\Lambda}_2^{\mathrm{T}} - \sum_{k=1}^{2} \mu_k \Psi_{11}^{(k)} & \star & \star \\ \begin{pmatrix} P \\ Q \\ \Lambda L^{\mathrm{T}} P \end{pmatrix}^{\mathrm{T}} - \sum_{k=1}^{2} \mu_k \Psi_{21}^{(k)} & -\sum_{k=1}^{2} \mu_k \Psi_{22}^{(k)} & \star \\ 0 & \begin{pmatrix} C & D & 0 \\ 0 & 0 & \frac{1}{2} I \end{pmatrix} \begin{pmatrix} P \\ Q \\ L^{\mathrm{T}} P \end{pmatrix} & -\begin{pmatrix} I & \star \\ 0 & \Gamma \end{pmatrix} \end{pmatrix} < 0, \tag{5.4}$$

$$\begin{pmatrix} P & \star \\ I & \gamma^2 R^{-1} \end{pmatrix} > 0.$$

Here, $\widehat{\Lambda}_2 = \mathrm{col}(0, 0, \Lambda_2)$, *and* $\Psi_{ij}^{(k)}$ *are the corresponding blocks of the matrices* $\Psi^{(1)}$ *and* $\Psi^{(2)}$ *given by* (3.7) *and* (3.15), *respectively.*

Proof of Theorem 5.1. By Theorem 4.1, system (5.1), (5.2) is absolutely stable and $J(\Theta) < \gamma^2$ if the generalized H_∞ norm from the input v to the output z of the linear system (4.11) with the weight matrix $\gamma^{-2} R$ is smaller than 1. In turn, this condition holds iff the generalized H_∞ norm of the dual system (4.12) is smaller than 1 (Lemma 4.2). Using the notations introduced above, we represent equations (4.12) as

$$\partial x_d(t) = \begin{pmatrix} I \\ \Theta \\ \Lambda L^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \left[\Delta^{\mathrm{T}} x_d(t) + \begin{pmatrix} C^{\mathrm{T}} & 0 \\ D^{\mathrm{T}} & 0 \\ 0 & \frac{1}{2} \Lambda^{-1} \Gamma^{-1/2} \end{pmatrix} v_d(t) \right], \tag{5.5}$$

$$z_d(t) = \Gamma^{1/2} \widehat{\Lambda}_2^{\mathrm{T}} \Delta^{\mathrm{T}} x_d(t).$$

Consider an augmented system with the additional artificial input $w_\Delta(t) \in L_2$ and output $z_\Delta(t)$ described by the equations

$$\partial x_a(t) = \begin{pmatrix} I \\ \Theta \\ \Lambda L^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \left[w_\Delta(t) + \begin{pmatrix} C^{\mathrm{T}} & 0 \\ D^{\mathrm{T}} & 0 \\ 0 & \frac{1}{2}\Lambda^{-1}\Gamma^{-1/2} \end{pmatrix} v_a(t) \right], \tag{5.6}$$

$$z_a(t) = \Gamma^{1/2}\widehat{\Lambda}_2^{\mathrm{T}} w_\Delta(t), \quad z_\Delta(t) = x_a(t).$$

Note that for $w_\Delta(t) = \Delta^{\mathrm{T}} z_\Delta(t)$, equations (5.6) coincide with (5.5). Suppose that for all $t \geqslant 0$, the additional input and output signals in system (5.6) satisfy the two inequalities

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi^{(1)} \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} \leqslant 0, \quad \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi^{(2)} \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} \leqslant 0, \tag{5.7}$$

where the matrices $\Psi^{(1)}$ and $\Psi^{(2)}$ are given by (3.7) and (3.15). Let $\mathbf{W}_{\boldsymbol{\Delta}}$ denote the set of all such signals $w_\Delta(t)$. Given $w_\Delta(t) = \Delta^{\mathrm{T}} z_\Delta(t)$, for all $\Delta \in \boldsymbol{\Delta}$, from (3.6) and (3.14) it follows that

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi^{(1)} \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} = z_\Delta^{\mathrm{T}}(t) \begin{pmatrix} \Delta^{\mathrm{T}} \\ I \end{pmatrix}^{\mathrm{T}} \Psi^{(1)} \begin{pmatrix} \Delta^{\mathrm{T}} \\ I \end{pmatrix} z_\Delta(t) \leqslant 0,$$

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi^{(2)} \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} = z_\Delta^{\mathrm{T}}(t) \begin{pmatrix} \Delta^{\mathrm{T}} \\ I \end{pmatrix}^{\mathrm{T}} \Psi^{(2)} \begin{pmatrix} \Delta^{\mathrm{T}} \\ I \end{pmatrix} z_\Delta(t) \leqslant 0.$$

Thus, $w_\Delta(t) = \Delta^{\mathrm{T}} z_\Delta(t) \in \mathbf{W}_{\boldsymbol{\Delta}}$ and, consequently, for $\Delta \in \boldsymbol{\Delta}$ system (5.5) is "immersed" in the augmented system (5.6), (5.7).

The proof below is for the continuous-time case: in the discrete-time one, it can be repeated by analogy. We establish conditions for the existence of a positive definite quadratic function $V(x_a) = x_a^{\mathrm{T}} P x_a$ with $P > \gamma^{-2} R$ that satisfies, for all x_a and v_a ($|x_a|^2 + |v_a|^2 \neq 0$), the inequality

$$\dot{V} + |z_a|^2 - |v_a|^2 < 0 \tag{5.8}$$

along the trajectories of the augmented system (5.6) for all $w_\Delta(t)$ with (5.7).

By the S-procedure, a sufficient condition is the existence of a function $V(x_a) = x_a^{\mathrm{T}} P x_a$ with $P > \gamma^{-2} R$ that satisfies, for all x_a, v_a, and w_Δ ($|x_a|^2 + |v_a|^2 + |w_\Delta|^2 \neq 0$) and some $\mu_1, \mu_2 \geqslant 0$, the inequality

$$\dot{V} + |z_a|^2 - |v_a|^2 - \sum_{k=1}^{2} \mu_k \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix}^{\mathrm{T}} \Psi^{(k)} \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix} < 0 \tag{5.9}$$

along the trajectories of the augmented system (5.6).

This inequality reduces to an inequality for a quadratic form in the variables w_Δ, x_a, and v_a with the matrix

$$\begin{pmatrix} \widehat{\Lambda}_2 \Gamma \widehat{\Lambda}_2^{\mathrm{T}} - \sum_{k=1}^{2} \mu_k \Psi_{11}^{(k)} & \star & \star \\ \begin{pmatrix} P \\ Q \\ \Lambda L^{\mathrm{T}} P \end{pmatrix}^{\mathrm{T}} - \sum_{k=1}^{2} \mu_k \Psi_{21}^{(k)} & -\sum_{k=1}^{2} \mu_k \Psi_{22}^{(k)} & \star \\ 0 & \begin{pmatrix} I & 0 \\ 0 & \Gamma^{-1/2} \end{pmatrix} \begin{pmatrix} C & D & 0 \\ 0 & 0 & \frac{1}{2}\Lambda^{-1} \end{pmatrix} \begin{pmatrix} P \\ Q \\ \Lambda L^{\mathrm{T}} P \end{pmatrix} & -I \end{pmatrix}.$$

Multiplying this matrix on the left and right by $\mathrm{diag}\left(I, I, \begin{pmatrix} I & 0 \\ 0 & \Gamma^{1/2} \end{pmatrix}\right)$ gives the matrix in the left-hand side of (5.4). For $\Delta \in \boldsymbol{\Delta}$, system (5.5) is immersed into the augmented system (5.6), (5.7). Therefore, we have the inequality $\dot{V} + |z_d|^2 - |v_d|^2 < 0$ for system (5.5) for all $\Delta \in \boldsymbol{\Delta}$, and its generalized H_∞ norm with the weight matrix $\gamma^{-2}R$ is smaller than 1. The proof of this theorem is complete.

Remark 3. According to the lossless S-procedure under two quadratic constraints (Theorem 4.1 in [13]), if $\mu_1\Psi^{(1)} + \mu_2\Psi^{(2)} > 0$ for some μ_1 and μ_2 (this LMI can be directly solved with respect to μ_1 and μ_2 after forming the particular matrix $\Psi^{(1)}$), then the corresponding inequality (5.9) is a sufficient and also necessary condition for the existence of the above function $V_a(x_a) = x_a^{\mathrm{T}} P x_a$ for the augmented system.

Remark 4. In the control design procedure based on only experimental data or only a priori information, due to the lossless S-procedure with one constraint, the conditions of Theorem 5.1 are sufficient and also necessary to fulfill inequality (5.8) along the trajectories of the augmented system (5.6), (5.7).

We denote by γ_*, γ_a, and γ_p the minimum upper bounds for the performance index $J(\Theta)$ that can be achieved using the control laws designed from experimental data and a priori information, only from a priori information, and only from experimental data, respectively, by Theorem 5.1. These bounds will be called guaranteed. The minimum values of γ for which inequalities (5.4) are solvable under $\mu_k \geqslant 0$, $k = 1, 2$, do not exceed, first, the minimum values of γ under $\mu_1 \equiv 0$ and $\mu_2 \geqslant 0$ and, second, the minimum values of γ under $\mu_1 \geqslant 0$ and $\mu_2 \equiv 0$. Therefore, Theorem 5.1 directly leads to the inequality

$$\gamma_* \leqslant \min\{\gamma_a, \gamma_p\},$$

which explains the advantage of control laws based on both a priori information and experimental data over those based on only a priori information or only experimental data. On the one hand, given rough a priori information (i.e., when the radius ρ of the matrix sphere in (3.13) is large enough and, accordingly, γ_a takes a high value), the index γ_* may turn out to be small if the measurement noise is not very significant (i.e., if the matrix ellipsoid $\boldsymbol{\Delta}_\mathbf{p}$ is small). On the other hand, if the measurement noise turns out to be significant and, accordingly, γ_p is large (furthermore, if the information matrix is singular and the matrix ellipse $\boldsymbol{\Delta}_\mathbf{p}$ is unbounded), then γ_* can nevertheless become small due to the smallness of the radii of the matrix sphere when using the a priori information. These conclusions will be confirmed by the simulation results in Section 6.

6. AN ILLUSTRATIVE EXAMPLE: A NONLINEAR OSCILLATOR

Let us design an absolutely stabilizing control law for the discrete-time model

$$x(t+1) = \begin{pmatrix} 1 & h \\ 0 & 1-\delta h \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ h \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ -h\omega^2 \end{pmatrix} \varphi(y(t)) + \begin{pmatrix} 0 \\ h \end{pmatrix} w(t),$$

$$y(t) = (1\ 0)x(t), \quad z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} u$$

of the nonlinear system

$$\ddot{\psi} + \delta\dot{\psi} + \omega^2\varphi(\psi) = u + w,$$

where $x = \mathrm{col}(\psi, \dot{\psi})$ and the nonlinearity $\varphi(\psi)$ satisfies condition (2.3) for $\alpha = -2/3\pi$ and $\beta = 1$. When implementing the control design procedure for a continuous-time system, it is necessary to calculate the derivatives, which entails an additional disturbance with bounds difficult to estimate in

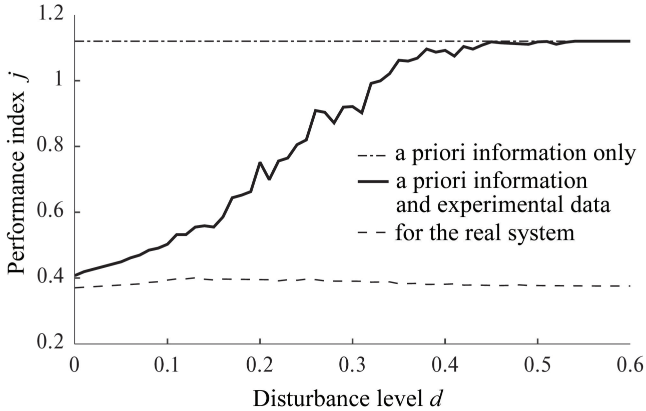

Fig. 2. The guaranteed values of the performance index under the robust controller based on experimental data and a priori information and its values under the same controller for the real system as functions of the disturbance level.

advance. In this sense, the discrete-time one is more preferable. In the experiment, it was assumed that the real system is a nonlinear oscillator with $\varphi(\psi) = \sin\psi$, a damping coefficient of $\delta = 0.1$, and a frequency of $\omega^2 = 1$; for the nominal system, $\delta_* = 0$ and $\omega_*^2 = 0.8$. The step $h = 0.2$, the weight matrix $R = 0.1I$, and the a priori uncertainty radius $\rho = 0.05$ were chosen. For each value of the disturbance level d, ten measurements were carried out, i.e., $N = 10$. In the experiment, the initial conditions and control were selected random in the interval $[-1, 1]$ and the disturbance $w(t)$ random in the interval $[-d, d]$. Inequalities (5.3) were solved using the CVX package, with "a slight departure from zero" to solve strict inequalities.

The figures demonstrate the simulation results obtained by averaging over 20 independent experiments. The following conclusions can be drawn from Fig. 2. As the disturbance level grows, the guaranteed value γ_* of the performance index calculated from the experimental data and a priori information increases, while remaining much smaller (under relatively small disturbance levels) than its counterpart γ_a obtained based on only the a priori information. The explanation is that the set $\mathbf{\Delta_p}$ of all systems consistent with the experimental data expands when increasing the disturbance level. Starting from some disturbance level (in the current example, approximately $d = 0.5$), the set $\mathbf{\Delta_p}$ includes the set $\mathbf{\Delta_a}$ of all systems separated based on the a priori information; therefore, with further increase of the disturbance level, γ_* stops growing and $\gamma_* = \gamma_a$. The dotted curve in Fig. 2 corresponds to the values of the performance index for the real system (if it were known) under the obtained robust controller based on the experimental data and a priori information under different disturbance levels. According to the experiments, this value weakly depends on the disturbance level and is close enough to the optimal value for the known system, i.e., $\gamma^2 \approx 0.39$.

The optimal controller and the corresponding value of the performance index for the real system (if it were known), computed by solving the LMIs (4.13), are as follows:

$$u = -6.58x_1 - 4.59x_2, \quad \gamma^2 = 0.36.$$

The robust controller and the corresponding value of the performance index, computed only from the a priori information using the LMIs (5.3) with $\mu_1 = 0$, are as follows:

$$u = -6.45x_1 - 5.38x_2, \quad \gamma_a^2 = 1.12.$$

The robust controller and the corresponding value of the performance index computed from the a priori information and experimental data in one of the experiments with the disturbance level $d = 0.1$ using the LMIs (5.3) with $\mu_1 \geqslant 0$ and $\mu_2 \geqslant 0$ are as follows:

$$u = -9.35x_1 - 6.61x_2, \quad \gamma_*^2 = 0.52.$$

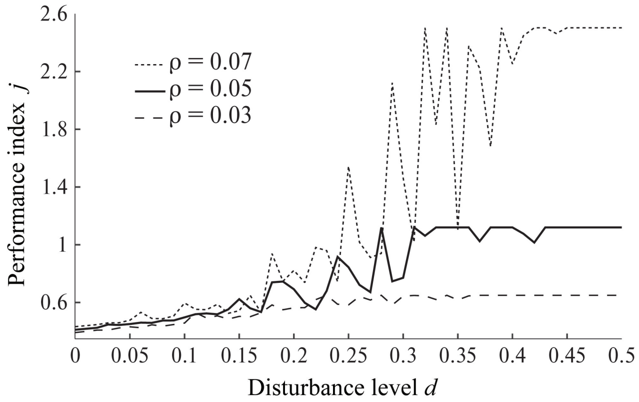

Fig. 3. The guaranteed values of the performance index under the robust controller based on experimental data and a priori information with different radii of matrix spheres as functions of the disturbance level.

As shown by the experiment, if the robust control law is built only from experimental data without a priori information (i.e., it is calculated using the LMIs (5.3) with $\mu_2 = 0$), then very large guaranteed values of the performance index are observed even under relatively small disturbance levels. The explanation is that even for small random disturbances the ellipsoid $\mathbf{\Delta_p}$ can be large enough or even degenerate (i.e., unbounded). Thus, considering a priori information has a regularizing effect on the robust control design procedure based on experimental data, even in the case when there is no unique robust controller on the entire set $\mathbf{\Delta_a}$ of systems identified from a priori information.

Figure 3 illustrates how increasing the radius of the matrix sphere in the a priori information affects the guaranteed value of the performance index under the robust controller obtained from the a priori information and experimental data.

7. CONCLUSIONS

In this paper, we have developed a method for designing an absolutely stabilizing control law for unknown Lurie systems that ensures the guaranteed value of the integral quadratic performance index characterizing the closed-loop system transients under uncertain initial conditions. The experimental data are not subject to the persistent excitation condition of the system, which is necessary for its identifiability. The resulting LMIs for calculating the feedback parameters serve to find the control laws based on only a priori information, only experimental data, and both a priori information and experimental data. The simulation results of experiments with a nonlinear oscillator have confirmed the advantage of control laws designed from experimental data and a priori information over those obtained using only experimental data or only a priori information.

FUNDING

This work was supported by the Russian Science Foundation, project no. 24-11-20023.

APPENDIX

Proof of Lemma 3.1. We write inequality (3.5) as

$$\Delta\widehat{X}\widehat{X}^{\mathrm{T}}\Delta^{\mathrm{T}} - X_+\widehat{X}^{\mathrm{T}}\Delta^{\mathrm{T}} - \Delta\widehat{X}X_+^{\mathrm{T}} + X_+X_+^{\mathrm{T}} - \Omega \leqslant 0.$$

With the change of variables, it becomes

$$\widehat{\Delta}^{(1)}\Sigma^2\widehat{\Delta}^{(1)\mathrm{T}} - X_+\widehat{X}^{(1)\mathrm{T}}\widehat{\Delta}^{(1)\mathrm{T}} - \widehat{\Delta}^{(1)}\widehat{X}^{(1)}X_+^{\mathrm{T}} + X_+X_+^{\mathrm{T}} - \Omega \leqslant 0.$$

Completing the square yields

$$[\widehat{\Delta}^{(1)} - X_+\widehat{X}^{(1)\mathrm{T}}\Sigma^{-2}]\Sigma^2[\widehat{\Delta}^{(1)} - X_+\widehat{X}^{(1)\mathrm{T}}\Sigma^{-2}]^{\mathrm{T}} \leqslant \Gamma,$$

where Γ is given by (3.11). Due to the expression for X_+ (3.9) and $\widehat{X}^{(1)}\widehat{X}^{(1)\mathrm{T}} = \Sigma^2$, we obtain $\Gamma = \Omega + W(\widehat{X}^{(1)\mathrm{T}}\Sigma^{-2}\widehat{X}^{(1)} - I)W^{\mathrm{T}}$. In view of (3.4), it follows that $\Gamma \geqslant 0$. Consider the matrix norm of the residual, i.e., the function $\mathrm{tr}\,(X_+ - \widehat{\Delta}^{(1)}\widehat{X}^{(1)})^{\mathrm{T}}(X_+ - \widehat{\Delta}^{(1)}\widehat{X}^{(1)})$. Equating its gradient with respect to $\widehat{\Delta}^{(1)}$ to zero, $-2X_+\widehat{X}^{(1)\mathrm{T}} + 2\widehat{\Delta}^{(1)}\widehat{X}^{(1)}\widehat{X}^{(1)\mathrm{T}} = 0$, we finally get the least-squares estimate $\Delta^{(1)}_{LS}$ of the unknown matrix $\Delta^{(1)}_{real}$ in (3.9) in the form $\widehat{\Delta}^{(1)}_{LS} = X_+\widehat{X}^{(1)\mathrm{T}}\Sigma^{-2}$.

Proof of Lemma 4.1. Due to $\varphi_i(y_i,t)[\varphi_i(y_i,t) - y_i] \leqslant 0$, letting $v = \varphi(y,t)$ in (4.6) yields the inequality $\triangle V + |z|^2 < 0$ along the trajectories of system (4.4), (4.5). In other words, $\dot{V} + |z|^2 < 0$ and, consequently, $\lim_{t\to\infty} x(t) = 0$. Since $Y < \gamma^2 R^{-1}$, after summation and integration we finally arrive at $\|z\|^2 < \gamma^2 x_0^{\mathrm{T}} R^{-1} x_0$.

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This paper was recommended for publication by N.V. Kuznetsov, a member of the Editorial Board